On the Asymptotic Evaluation of $\int_0^{\pi/2} J_0^2(\lambda \sin x)\, dx$

By Basil J. Stoyanov and Richard A. Farrell

Abstract. The asymptotic behavior of the integral

$$I(\lambda) = \int_0^{\pi/2} J_0^2(\lambda \sin x)\, dx$$

is investigated, where $J_0(x)$ is the zeroth-order Bessel function of the first kind and λ is a large positive parameter. A practical analytical expression of the integral at large λ is obtained and the leading term is $(\ln \lambda)/(\lambda \pi)$.

Recently, while working on a variational formulation of diffraction theory, we encountered the following integral involving the zeroth-order Bessel function

(1) $$I(\lambda) = \int_0^{\pi/2} J_0^2(\lambda \sin x)\, dx.$$

The asymptotic behavior of this integral for large values of the parameter λ was of particular interest.

It is a straightforward matter to express this integral as the infinite series

(2a) $$I(\lambda) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{\left[\Gamma\left(n + \frac{1}{2}\right)\right]^2}{(n!)^4} \left(-\lambda^2\right)^n$$

by using the well-known [9] series representation of $J_0^2(z)$, or as

(2b) $$I(\lambda) = \frac{\pi}{2} \sum_{n=0}^{\infty} \varepsilon_n J_n^4\left(\frac{\lambda}{2}\right), \quad \text{where } \varepsilon_n = \begin{cases} 1, & n = 0, \\ 2, & n = 1, 2, \ldots \end{cases}$$

by using the integral representation of $J_0(z)$ together with Neumann's addition theorem [1]. The former result can be found as a particular case of [7, Eq. (10.46.1)]. We also note that Eq. (2a) is $(\pi/2)$ times the series representation of the generalized hypergeometric function $_2F_3(\frac{1}{2}, \frac{1}{2}; 1, 1, 1; -\lambda^2)$ (see, e.g., [2, p. 240, Eq. (26)] or [7, Eq. (10.48.1)]).

However, these expressions cannot readily be transformed into representative asymptotic (for large λ) series and are of little value when λ is large enough because so many terms are required to evaluate the sums. An instructive simple procedure which provides a practical analytical expression of the integral at large λ is

described below.* Briefly, we demonstrate that, for large λ,

$$I(\lambda) = (\ln \lambda + 4\ln 2 + \gamma)/(\lambda\pi) + \sin(2\lambda - \pi/4)/(2\sqrt{\pi}\,\lambda^{3/2}) + O(\lambda^{-5/2}),$$

with γ being Euler's constant, $0.57721\ldots$. The asymptotic approximation is tested against numerical evaluation of the integral and the agreement is quite good, being to about 0.3% at $\lambda = 10$ and 0.006% at $\lambda = 50$.

Our proof of this asymptotic formula begins by recognizing that, except for a tiny region in the neighborhood of $x = 0$, the Bessel function can be approximated by its asymptotic value. Thus, introducing the new variable $t = \sin x$, we rewrite the integral as

$$(3) \qquad I(\lambda) = \int_0^{B/\lambda} \frac{J_0^2(\lambda t)}{\sqrt{1 - t^2}}\, dt + \int_{B/\lambda}^1 \frac{J_0^2(\lambda t)}{\sqrt{1 - t^2}}\, dt \equiv I_1(\lambda) + I_2(\lambda),$$

where B is a large number which is, however, small compared to λ, so that $B/\lambda \ll 1$. For example, we can set $B = \lambda^\varepsilon$ with $0 < \varepsilon < 1$. Of course, the final result should be independent of auxiliary parameters, such as ε, introduced in intermediate calculations. However, it will be convenient to choose $\varepsilon = 1/2$ for the purposes of order-of-magnitude bookkeeping. We will explicitly demonstrate that the neglected terms are at least of order λ^{-2}. In fact, however, lengthy algebraic manipulations show that the neglected terms are actually of order $\lambda^{-5/2}$ (and smaller). Although this algebra will not be presented, the order of the largest neglected terms will be indicated at each step.

In the first integral in (3) (denoted by $I_1(\lambda)$), we can use an approximation $1/\sqrt{1 - t^2} = 1 + t^2/2 + O(t^4)$ and with the substitution $y = \lambda t$,

$$(4) \qquad I_1(\lambda) = \frac{1}{\lambda} \int_0^B J_0^2(y)\, dy + \frac{1}{2\lambda^3} \int_0^B y^2 J_0^2(y)\, dy + O(\lambda^{-3}),$$

where the last integral (as well as those neglected) can be expressed in terms of the first one [8, p. 256, Eq. (17)]. In this way one finds that the contribution from the last integral is of order λ^{-2}, and it will not be evaluated explicitly in this paper. Because B is large, $J_0(y)$ does not differ significantly from $J_0(y(1 + B^{-(1+\alpha)})) \equiv J_0(y(1 + \delta))$ in the interval $0 \leqslant y \leqslant B$, as can be readily demonstrated from Neumann's addition theorem, where it has been assumed that the parameter α is greater than zero and, for convenience, will be chosen to be large (of order λ) while estimating the correction terms in $I_1(\lambda)$. Under these conditions,

$$I_1(\lambda) = \frac{1}{\lambda} \int_0^B J_0[y(1 + \delta)] J_0(y)\, dy + O(\lambda^{-2})$$

$$(5) \qquad = \frac{1}{\lambda} \left[\int_0^\infty J_0[y(1 + \delta)] J_0(y)\, dy - \int_B^\infty J_0[y(1 + \delta)] J_0(y)\, dy \right]$$

$$+ O(\lambda^{-2}).$$

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*The authors of the book [3] discuss general methods for obtaining asymptotic expansions of integrals. We would like to thank the referee of our paper for outlining the derivation of an alternative procedure for evaluating Eq. (1) based on the Mellin transform techniques of that book. We find that this procedure also leads to Eq. (16), including the fact that the next-order correction terms are $O(\lambda^{-5/2})$.

The penultimate integral is a table integral (see [6, Eq. (6.512-1)]),

(6)
$$\int_0^\infty J_0[y(1+\delta)]\, J_0(y)\, dy = \frac{1}{(1+\delta)}\, {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{1}{(1+\delta)^2}\right)$$

$$= \frac{1}{\pi}\left[3\ln 2 + (1+\alpha)\ln B\right] + O(\lambda^{-(1+\alpha)/2}\ln\lambda),$$

where we have used the behavior of the hypergeometric function when the variable is approximately unity (see, e.g., [1, Eq. (15.3.10)]), and recalled that $\delta = 1/B^{1+\alpha}$.

In the last integral in (5) we can employ the asymptotic expression of $J_0(z)$ for large z (see, e.g., [9, p. 194]) to obtain

(7)
$$\int_B^\infty J_0[y(1+\delta)]\, J_0(y)\, dy$$

$$= \frac{1}{\pi\sqrt{1+\delta}}\left(\int_{(2+\delta)B}^\infty \frac{\sin\xi}{\xi}\, d\xi + \int_{\delta B}^\infty \frac{\cos\xi}{\xi}\, d\xi\right) + O(\lambda^{-1})$$

$$= \frac{1}{\pi\sqrt{1+\delta}}\left[-\mathrm{si}((2+\delta)B) - \mathrm{Ci}(\delta B)\right] + O(\lambda^{-1})$$

$$= \frac{-1}{\pi}\left[\gamma - \alpha\ln B - \frac{\cos(2B)}{2B}\right] + O(\lambda^{-1}),$$

where the behavior [1] of the sine and cosine integrals at respectively large and small z was utilized. Therefore, substituting (6) and (7) into (5), we get

(8)
$$I_1(\lambda) = \frac{1}{\lambda\pi}\left[3\ln 2 + \gamma + \ln B - \frac{\cos(2B)}{2B}\right] + O(\lambda^{-2}).$$

Again with the asymptotic expression of $J_0(z)$ for large z, the second integral in (3) (denoted by $I_2(\lambda)$) yields

(9)
$$I_2(\lambda) = \int_{B/\lambda}^1 \frac{J_0^2(\lambda t)}{\sqrt{1-t^2}}\, dt$$

$$= \frac{1}{\lambda\pi}\left[\int_{B/\lambda}^1 \frac{dt}{t\sqrt{1-t^2}} + \int_{B/\lambda}^1 \frac{\sin(2\lambda t)}{t\sqrt{1-t^2}}\, dt\right] + O(\lambda^{-2}),$$

where [4]

(10)
$$\int_{B/\lambda}^1 \frac{dt}{t\sqrt{1-t^2}} = (\ln\lambda + \ln 2 - \ln B) + O(\lambda^{-1}),$$

while

(11)
$$\int_{B/\lambda}^1 \frac{\sin(2\lambda t)}{t\sqrt{1-t^2}}\, dt = -\int_0^{B/\lambda} \frac{\sin(2\lambda t)}{t\sqrt{1-t^2}}\, dt + \int_0^1 \frac{\sin(2\lambda t)}{t\sqrt{1-t^2}}\, dt.$$

For the penultimate integral, $0 \leqslant t \leqslant B/\lambda \ll 1$ and $1/\sqrt{1-t^2} = 1 + O(t^2)$, so that with the substitution $y = 2\lambda t$, we have [1]

(12)
$$\int_0^{B/\lambda} \frac{\sin(2\lambda t)}{t\sqrt{1-t^2}}\, dt = \int_0^{2B} \frac{\sin y}{y}\, dy + O(\lambda^{-3/2})$$

$$= \frac{\pi}{2} - \frac{\cos(2B)}{2B} + O(\lambda^{-1}).$$

Finally, the last integral in (11) can be rewritten as (see [6, Eq. (3.753-2)])

$$
(13) \quad \int_0^1 \frac{dt}{\sqrt{1 - t^2}} \int_0^{2\lambda} \cos(yt)\, dy = \frac{\pi}{2} \int_0^{2\lambda} J_0(y)\, dy
$$

$$
= \frac{\pi}{2} \left\{ 2\lambda J_0(2\lambda) + \lambda\pi \left[\mathbf{H}_0(2\lambda) J_1(2\lambda) - \mathbf{H}_1(2\lambda) J_0(2\lambda) \right] \right\},
$$

where [6, Eq. (6.511-6)] was used to express the result in terms of the Struve functions $\mathbf{H}_0$ and $\mathbf{H}_1$. Since $\lambda \gg 1$, these functions can be approximated by [1]

$$
\mathbf{H}_0(2\lambda) = Y_0(2\lambda) + \frac{1}{\lambda\pi} + O(\lambda^{-3}), \qquad \mathbf{H}_1(2\lambda) = Y_1(2\lambda) + \frac{2}{\pi} + O(\lambda^{-2})
$$

and, taking into account the Wronskian [1]

$$
Y_0(2\lambda) J_1(2\lambda) - Y_1(2\lambda) J_0(2\lambda) = (\lambda\pi)^{-1},
$$

we arrive at

$$
(14) \quad \int_0^1 \frac{\sin(2\lambda t)}{t\sqrt{1 - t^2}}\, dt = \frac{\pi}{2} \left[1 + \frac{\sin(2\lambda - \pi/4)}{\sqrt{\lambda\pi}} \right] + O(\lambda^{-3/2}),
$$

where the discarded correction terms are independent of the auxiliary parameters. We note in passing that the asymptotic techniques developed by Erdélyi [5, Section 2.8] provide an alternative procedure for evaluating the integral on the left side of Eq. (11).

From Eqs. (9)–(12) and (14) it follows that

$$
(15) \quad \begin{aligned} I_2(\lambda) &= \frac{1}{\lambda\pi} \left[\ln\lambda + \ln 2 - \ln B + \frac{\cos(2B)}{2B} \right] \\ &\quad + \frac{\sin(2\lambda - \pi/4)}{2\sqrt{\pi}\,\lambda^{3/2}} + O(\lambda^{-2}), \end{aligned}
$$

while inserting (8) and (15) into (3) we obtain (after some cancellations among the neglected correction terms)

$$
(16) \quad \int_0^{\pi/2} J_0^2(\lambda \sin x)\, dx = \frac{1}{\lambda\pi} (\ln\lambda + 4\ln 2 + \gamma)
$$

$$
+ \frac{\sin(2\lambda - \pi/4)}{2\sqrt{\pi}\,\lambda^{3/2}} + O(\lambda^{-5/2}).
$$

This result, comprising the leading and first correction terms of the asymptotic expansion of the integral, provides the desired analytical expression at large λ. It may be of interest to note that the correction term of order $\lambda^{-3/2}$ in Eq. (16) represents the stationary phase point (at $x = \pi/2$) contribution to the integral. The integral in Eq. (16) was numerically evaluated for comparison with the asymptotic formula. For values of $\lambda \gtrsim 5000$ this was accomplished using Eq. (2b). The asymptotic formula produces about three significant digits at $\lambda = 10$, which gradually improves to eight significant digits at $\lambda = 20{,}000$.

The Johns Hopkins University Applied Physics Laboratory
Johns Hopkins Road
Laurel, Maryland 20707

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